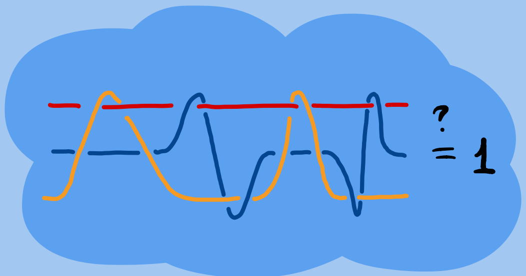
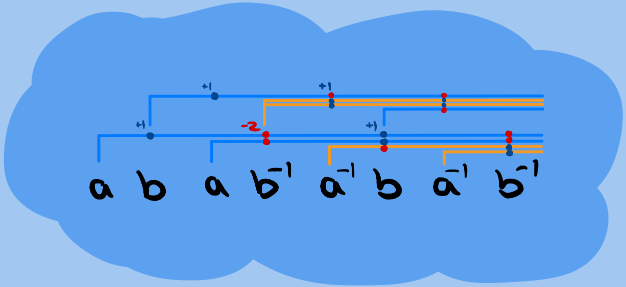
Letter-Braiding: Bridging group theory & Cohomology.

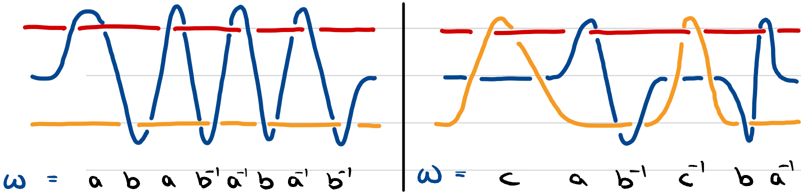




Nir Gadish

U. of Michigan

Goal: invariants of words in groups, detecting nontrivial elts.



$\omega = a\ b\ a\ b^{-1}\ a^{-1}\ b\ a^{-1}\ b^{-1}$

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$\omega \overset{?}{=} 1$

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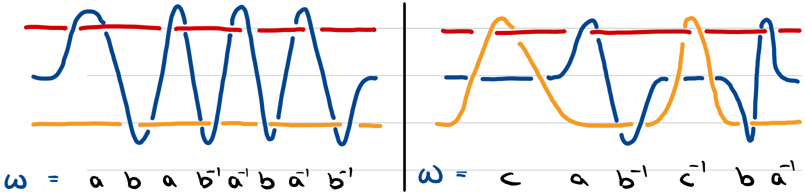

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Presentation $\pi = \langle \underset{\text{gen}}{S} \mid \underset{\text{rel}}{R} \rangle$

- word $\omega = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n}$ $(\varepsilon_i = \pm 1)$
- Invariant ~ up to relations

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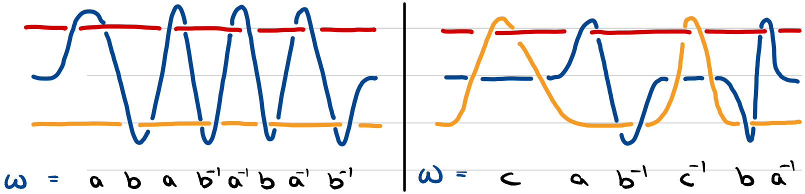

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Topological presentation

$\pi = \pi_1(X, *)$ $\left(X \begin{array}{l}\text{space}\\ \text{sSet}\end{array}\right)$

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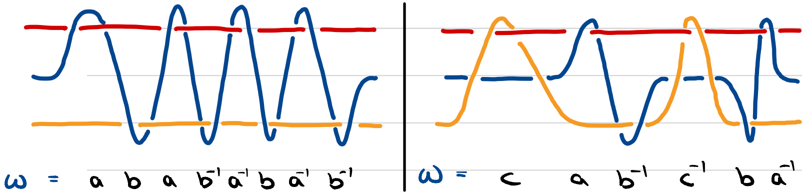

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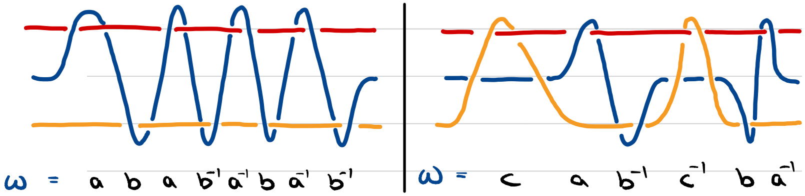

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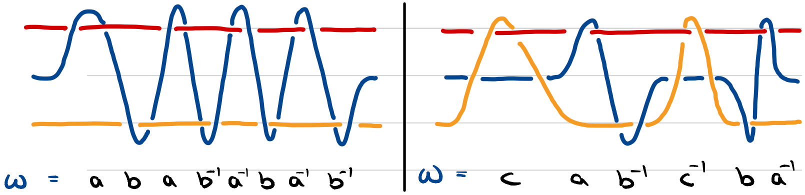

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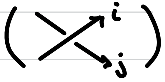

$\sim +1.$

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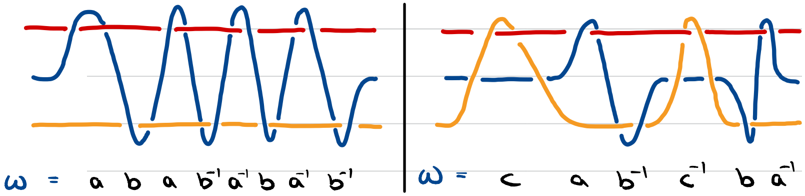

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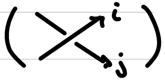

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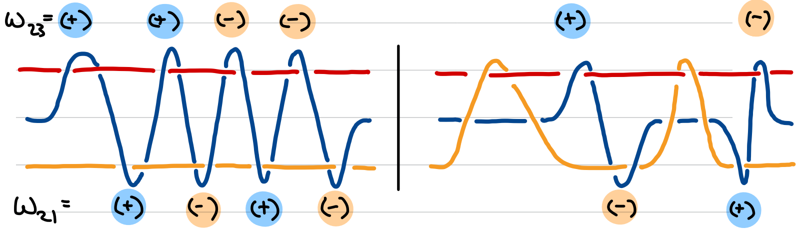

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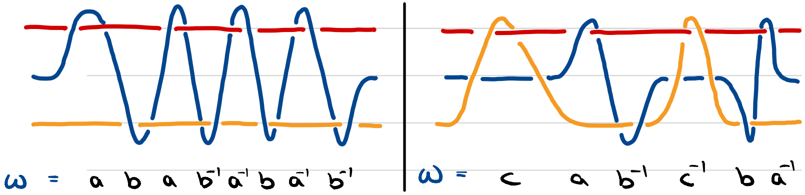

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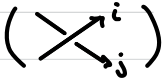

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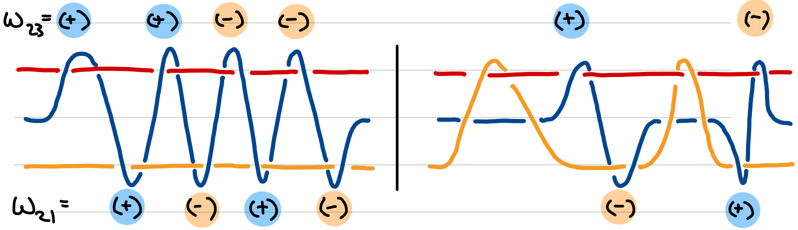

Hard: Vanish on $[\pi, \pi]$.

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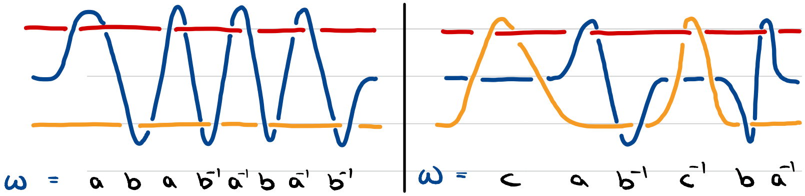

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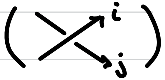
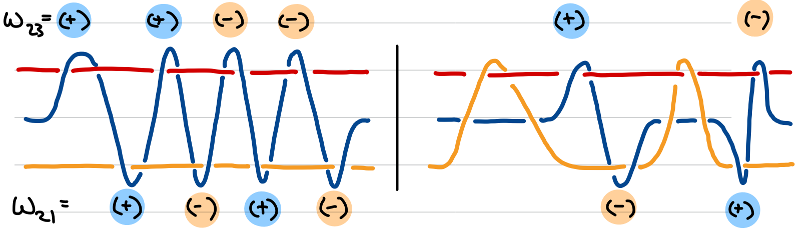
$\omega_{ij} = \#(i \text{ over } j)$ () ~ +1.



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Free groups [Magnus , Fox calculus]
30's 50's

Invariants detecting $[F,[\dots,[F,F]\dots]]$

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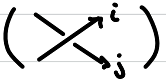

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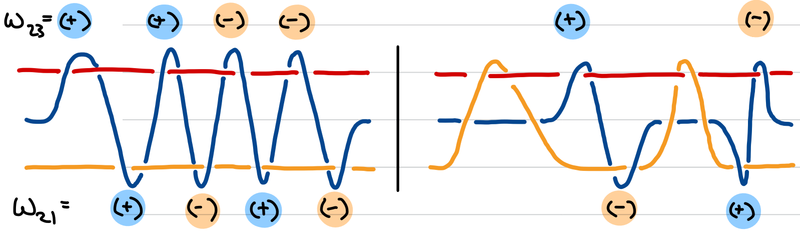

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Construction [G]. Letter braiding

$$F_n = \langle a, b, \dots \rangle$$

invariants: $A = \#\ a$'s
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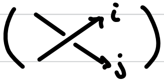

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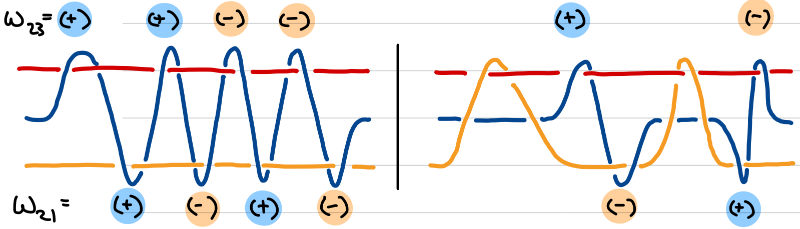

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see commutators

$A|B : a\; b\; a^{-1}\; b^{-1} \longmapsto +1.$

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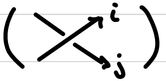

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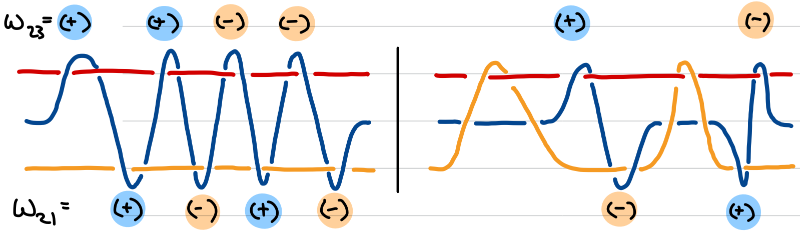

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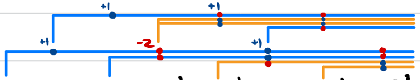

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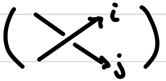

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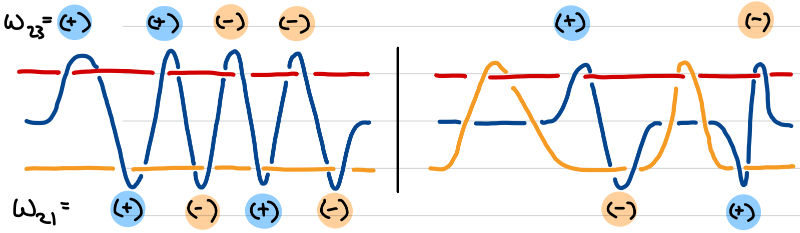

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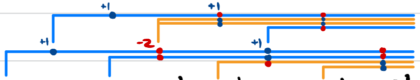

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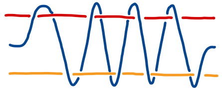
detects - $[a,[b,a]] =$



Invariants for free groups

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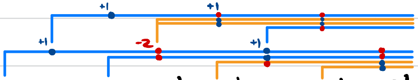

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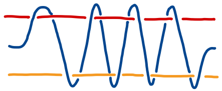

Applications

① $\pi_1(S^3 - (\underbrace{\bigcirc\ \bigcirc\ \bigcirc}_{\text{unlink}}))$ $\underline{\text{free}}$

$\Downarrow$

[Milnor] Link invariants

detecting Borromean rings

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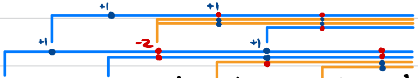

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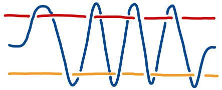

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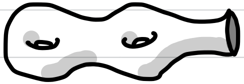
② $\pi_1($  $)$ free

$\Downarrow$

[Morita] Mapping class invariants

detecting Torelli, Johnson, ...

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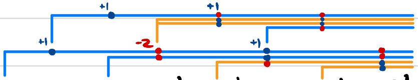

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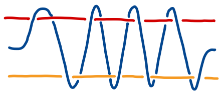

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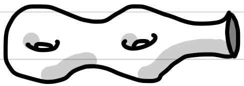
[Milnor] Link invariants

detecting Borromean rings

② $\pi_1($



$)$ free

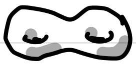
$\Downarrow$

[Morita] Mapping class invariants

detecting Torelli, Johnson, ...

Problem: relations ⇝ A|B not well-defined.

$\pi_1($



$) = \left\langle \begin{matrix} a,b \\ c,d \end{matrix} \;\middle|\; [a,b] = [c,d]^{-1} \right\rangle$

$A|B : [a,b] \longmapsto 1$

$A|B : [c,d]^{-1} \longmapsto 0$

$A|B$, $C|D$

✗ ✗

Invariants for free groups

Construction [G] Letter braiding

$F_n = \langle a, b, \dots \rangle$

invariants: $A = \#\, a\text{'s}$, $B = \#\, b\text{'s}$, $\vdots$ (with sign) $\Big\} \in H^1(F_n)$

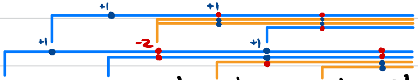
next level: $A|B = \#\,(a\text{'s } \underline{\text{before}}\ b\text{'s})$

$\vdots$

see commutators

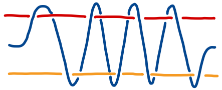
$\underline{A|B}$: $a\ b\ a^{-1}\ b^{-1} \longmapsto +1$.

iterate: $A_1 | A_2 | \dots | A_k = \#\,(a_1 \text{ before } a_2 \dots)$



$\underline{A|B|A}$: $a\ b\ \underbrace{a\ b^{-1}\ a^{-1}\ b}\ a^{-1}\ b^{-1} \longmapsto +2$

detects – $[a,[b,a]] =$



Applications

① $\pi_1(S^3 - (\bigcirc \bigcirc \bigcirc))$ $\underline{\text{free}}$

unlink

$\Downarrow$

[Milnor] Link invariants

detecting Borromean rings

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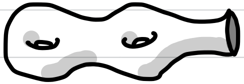

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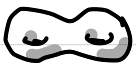

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$A|B : [c,d]^{-1} \longmapsto 0$ $\qquad$ ✗ $\quad$ ✗

But – $A|B - C|D$ ✓ , $A|B + B|A$ ✓ rule?!

Applications

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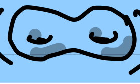
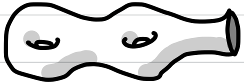
!! $\ker\big(H^1 \otimes H^1 \xrightarrow{\cup} H^2($  $)\big)$!!

$\Downarrow$

[Milnor] Link invariants

detecting Borromean rings

② $\pi_1($  $)$ free

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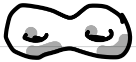
$\pi_1($  $) = \left\langle \begin{matrix} a,b \\ c,d \end{matrix} \,\middle|\, [a,b] = [c,d]^{-1} \right\rangle$

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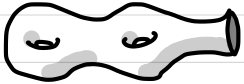
② $\pi_1($  $)$ free

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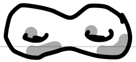
$\pi_1($  $) = \left\langle \substack{a,b\\c,d} \;\middle|\; [a,b]=[c,d]^{-1} \right\rangle$

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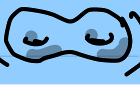
!! $\ker\left(H^1 \otimes H^1 \xrightarrow{\cup} H^2\right.($  $\left.)\right)$!!

Formalize w/ Bar constr.

$(C^\bullet, u, d)$ dg algebra (assoc. augmented)

$\Downarrow$

$\mathrm{Bar}(C^\bullet) = T(\tilde{C}^\bullet)$ double cplx

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① $\pi_1\left(S^3 - (\underbrace{\bigcirc\ \bigcirc\ \bigcirc}_{\text{unlink}})\right)$ **free**

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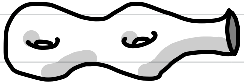
[Milnor] Link invariants

detecting Borromean rings

② $\pi_1\big($  $\big)$ **free**

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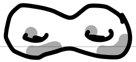
$\pi_1\big($  $\big) = \left\langle \begin{matrix} a, b \\ c, d \end{matrix} \,\middle|\, [a,b] = [c,d]^{-1} \right\rangle$

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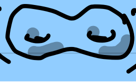
!! $\ker\left(H^1 \otimes H^1 \xrightarrow{\cup} H^2\big(\right.$  $\left.\big)\right)$!!

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0 & 0 & 0 & k \\
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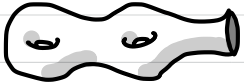
Applications

① $\pi_1\big(S^3 - (\underbrace{\bigcirc\ \bigcirc\ \bigcirc}_{\text{unlink}})\big)$ **free**

⇓

[Milnor] **Link invariants**

detecting Borromean rings

② $\pi_1\big($  $\big)$ **free**

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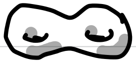
[Morita] **Mapping class invariants**

detecting Torelli, Johnson, ...

Problem: relations ⇝ $A|B$ not well-defined.

$\pi_1\big($  $\big) = \left\langle \begin{matrix} a,b \\ c,d \end{matrix} \,\middle|\, [a,b] = [c,d]^{-1} \right\rangle$

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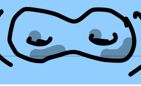
!! $\ker\big(H^1 \otimes H^1 \xrightarrow{\cup} H^2($  $)\big)$!!

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(deg 0) – $\mathrm{Bar}^0 = T(C^1)^{\otimes}$

Applications

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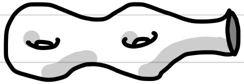
[Milnor] Link invariants

detecting Borromean rings

② $\pi_1($  $)$ free

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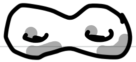
$\pi_1($  $) = \left\langle \begin{matrix} a, b \\ c, d \end{matrix} \;\middle|\; [a,b] = [c,d]^{-1} \right\rangle$

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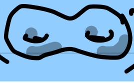
‼ $\ker\!\big(H^1 \otimes H^1 \xrightarrow{\cup} H^2($  $)\big)$ ‼

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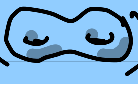
$X \rightsquigarrow C^\bullet_\Delta(X;k) \rightsquigarrow H^0_{\mathrm{Bar}}(C^\bullet_\Delta(X;k))$

sSet dga "Letter braiding inv."

here A|B|C

(deg 0) - $\mathrm{Bar}^0 = T(C^1)$

Ex. $H^1(X;k)$, $\ker(H^1 \otimes H^1 \xrightarrow{\cup} H^2(X))$, …

!! $\ker\left(H^1 \otimes H^1 \xrightarrow{\cup} H^2\left(\right.\right.$  $\left.\left.\right)\right)$!!

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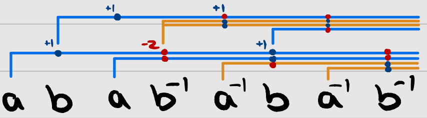


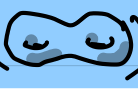
$\mapsto +2$

Thm. [C] k a PID ($\mathbb{Z}$, $\mathbb{F}_p$, ...)

Linear combination

$\sum A_{i_1}|\dots|A_{i_k} \in T(C^1_\Delta(X;k))$

$\rightsquigarrow$ function on "words" $\omega : S^1 \to X$.

!! $\ker\left(H^1\otimes H^1 \xrightarrow{\cup} H^2\left(\right.\right.$  $\left.\left.\right)\right)$!!

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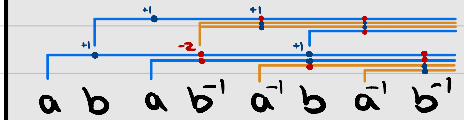

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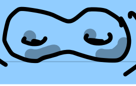
!! $\ker\left(H^1\otimes H^1 \xrightarrow{\cup} H^2\left(\right.\right.$  $\left.\left.\right)\right)$!!

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X ⇝ $C^\bullet_\Delta(X;k)$ ⇝ $H^0_{Bar}(C^\bullet_\Delta(X;k))$

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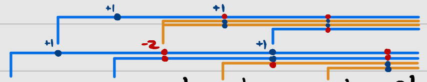

$a\ b\ a\ b^{-1}\ a^{-1}\ b\ a^{-1}\ b^{-1} \mapsto +2$

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⇕

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⇕

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[Complete "finite-type" invariant]

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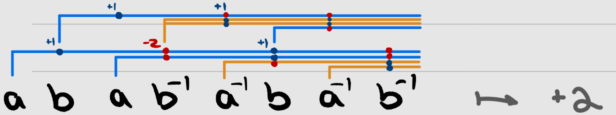
$A_1 | A_2 | \dots | A_k = \#(a_1 \text{ before } a_2 \dots)$



Thm. [G] k a PID $(\mathbb{Z}, \mathbb{F}_p, \dots)$

Linear combination

$$\sum A_{i_1} | \dots | A_{i_k} \in T(C^1_\Delta(X;k))$$

⇝ function on "words" $\omega : S^1 \to X$.

- well defined on $\pi = \pi_1(X, *)$

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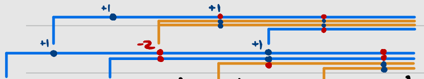

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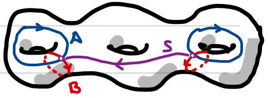

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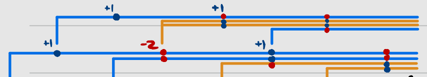

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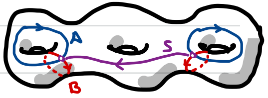

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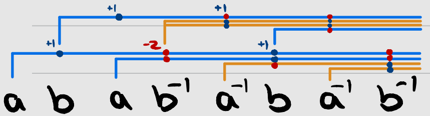

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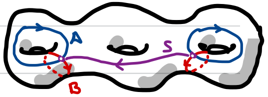

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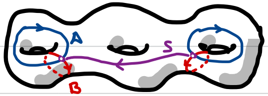

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Open: invariants for $\pi_1^{ét}$(Scheme)

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Need - inv. of deck trans. instead of $\omega: S^1 \to X$

or replace S^1 $\nearrow \mathbb{G}_m$ $\searrow \mathrm{Spec}\,\mathbb{F}_p$

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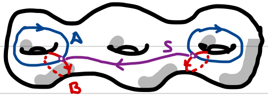

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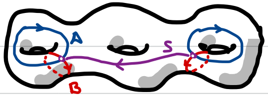

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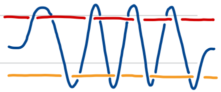
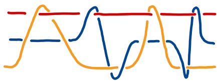


$\notin \gamma_4 PB_n$

Thank you !



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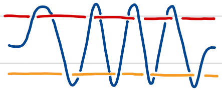

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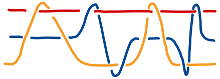

$= 1$

- Recall [Adams '56]

X simply connected,

$$\operatorname{Bar}(C^\bullet(X)) \simeq C^\bullet(\Omega X)$$

fails when $\pi_1(X) \neq 1$ (even nilpotent)

e.g. $k[\pi]/I^n$ depends on k.

- [Rivera - Zeinalian '18] X connected

$$\Omega(C_\bullet(X)) \simeq C_\bullet(\Omega X)$$

not effective

- not convergent
- solves word prob.

⟹ Undecidable!

$$\begin{tikzcd}
\mathbb{Z} & 0 & 0 & 0 \\
0 & C_1 \arrow[d] & 0 & 0 \\
0 & C_2 \arrow[r] \arrow[d] & C_1 \otimes C_1 \arrow[d] & 0 \\
0 & C_3 \arrow[r] \arrow[d] & \begin{matrix} C_1 \otimes C_2 \\ C_2 \otimes C_1 \end{matrix} \arrow[r] \arrow[d] & C_1 \otimes C_1 \otimes C_1 \arrow[d] \\
 & {} & {} & {}
\end{tikzcd}$$